SAT-based Explicit $LTL_f$ Satisfiability Checking

Jianwen Li, Kristin Y. Rozier
Iowa State University
Ames, IA, USA
{jianwen,kyrozier}@iastate.edu

Geguang Pu, Yueling Zhang
East China Normal University
Shanghai, China
{ggpu,ylzhang}@sei.ecnu.edu.cn

Moshe Y. Vardi
Rice University
Houston, TX, USA
vardi@cs.rice.edu

Abstract
We present here a SAT-based framework for $LTL_f$ (Linear Temporal Logic on Finite Traces) satisfiability checking. We use propositional SAT-solving techniques to construct a transition system for the input $LTL_f$ formula; satisfiability checking is then reduced to a path-search problem over this transition system. Furthermore, we introduce CDLSC (Conflict-Driven $LTL_f$ Satisfiability Checking), a novel algorithm that leverages information produced by propositional SAT solvers from both satisfiability and unsatisfiability results. Experimental evaluations show that CDLSC outperforms all other existing approaches for $LTL_f$ satisfiability checking, by demonstrating an approximate four-fold speed-up compared to the second-best solver.

Introduction
Linear Temporal Logic over Finite Traces, or $LTL_f$, is a formal language gaining popularity in the AI community for formalizing and validating system behaviors. While standard Linear Temporal Logic (LTL) is interpreted on infinite traces (Pnueli 1977), $LTL_f$ is interpreted over finite traces (De Giacomo and Vardi 2013). While LTL is typically used in formal-verification settings, where we are interested in nonterminating computations, cf. (Vardi 2007), $LTL_f$ is more attractive in AI scenarios focusing on finite behaviors, such as planning (Bacchus and Kabanza 1998; De Giacomo and Vardi 1999; Calvanese, De Giacomo, and Vardi 2002; Patrizi et al. 2011; Camacho et al. 2017), plan constraints (Bacchus and Kabanza 2000; Gabaldon 2004), and user preferences (Bienvenu, Fritz, and McIlraith 2006; 2011; Sohrabi, Baier, and McIlraith 2011). Due to the wide spectrum of applications of $LTL_f$ in the AI community (De Giacomo, Masellis, and Montali 2014), it is worthwhile to study and develop an efficient framework for solving $LTL_f$-reasoning problems. Just as propositional satisfiability checking is one of the most fundamental propositional reasoning tasks, $LTL_f$ satisfiability checking is a fundamental task for $LTL_f$ reasoning.

Given an $LTL_f$ formula, the satisfiability problem asks whether there is a finite trace that satisfies the formula. A "classical" solution to this problem is to reduce it to the LTL satisfiability problem (De Giacomo and Vardi 2013). The advantage of this approach is that the LTL satisfiability problem has been studied for at least a decade, and many mature tools are available, cf. (Rozier and Vardi 2007; 2010). Thus, $LTL_f$ satisfiability checking can benefit from progress in LTL satisfiability checking. There is, however, an inherent drawback that an extra cost has to be paid when checking $LTL_f$ formulas, as the tool searches for a "lasso" (a lasso consists of a finite path plus a cycle, representing an infinite trace), whereas models of $LTL_f$ formulas are just finite traces. Based on this motivation, (Li et al. 2014) presented a tableau-style algorithm for $LTL_f$ satisfiability checking. They showed that the dedicated tool, Aalta-finite, which conducts an explicit-state search for a satisfying trace, outperforms extant tools for $LTL_f$ satisfiability checking.

The conclusion of a dedicated solver being superior to $LTL_f$ satisfiability checking from (Li et al. 2014), seems to be out of date by now because of the recent dramatic improvement in propositional SAT solving, cf. (Malik and Zhang 2009). On one hand, SAT-based techniques have led to a significant improvement on LTL satisfiability checking, outperforming the tableau-based techniques of Aalta-finite (Li et al. 2014). (Also, the SAT-based tool ltl2sat for $LTL_f$ satisfiability checking outperforms Aalta-finite on particular benchmarks (Fionda and Greco 2016).) On the other hand, SAT-based techniques are now dominant in symbolic model checking (Cavada et al. 2014; Vizel, Weissenbacher, and Malik 2015). Our preliminary evaluation indicates that $LTL_f$ satisfiability checking via SAT-based model checking (Bradley 2011; Een, Mishchenko, and Brayton 2011) or via SAT-based LTL satisfiability checking (Li et al. 2015) both outperform the tableau-based tool Aalta-finite. Thus, the question raised initially in (Rozier and Vardi 2007) needs to be re-opened with respect to $LTL_f$ satisfiability checking: is it best to reduce to SAT-based model checking or develop a dedicated SAT-based tool?

Inspired by (Li et al. 2015), we present an explicit-state SAT-based framework for $LTL_f$ satisfiability. We construct the $LTL_f$ transition system by utilizing SAT solvers to compute the states explicitly. Furthermore, by making use of both satisfiability and unsatisfiability information from SAT solvers, we propose a conflict-driven algorithm, CDLSC, for efficient $LTL_f$ satisfiability checking. We show that by
specializing the transition-system approach of (Li et al. 2015) to LTL \_f and its finite-trace semantics, we get a framework that is significantly simpler and yields a much more efficient algorithm CDLSC than the one in (Li et al. 2015). We conduct a comprehensive comparison among different approaches. Our experimental results show that the performance of CDLSC dominates all other existing LTL \_f-satisfiability-checking algorithms. On average, CDLSC achieves an approximate four-fold speed-up, compared to the second-best solution (IC3 (Bradley 2011)+K-LIVE (Claessen and Sörensson 2012)) tested in our experiments. Our results re-affirm the conclusion of (Li et al. 2014) that the best approach to LTL \_f satisfiability solving is via a dedicated tool, based on explicit-state techniques.

\section*{LTL over Finite Traces}

Given a set \( \mathcal{P} \) of atomic propositions, an LTL \_f formula \( \phi \) has the form:

\[ \phi := tt \mid p \mid \neg \phi \mid \phi \land \phi \mid X\phi \mid \phi U\phi; \]

where \( tt \) is true, \( \neg \) is the negation operator, \( \land \) is the and operator, \( X \) is the strong Next operator and \( U \) is the Until operator. We also have the duals \( f f \) (false) for \( tt \lor \neg \land \), \( N \) (weak Next) for \( X \) and \( R \) for \( U \). A literal is an atom \( p \) in \( \mathcal{P} \) or its negation \( \neg p \). Moreover, we use the notation \( \mathcal{G} \phi \) (Globally) and \( \mathcal{F} \phi \) (Eventually) to represent \( f f R \phi \) and \( tt U \phi \). Notably, \( X \) is the strong next operator, while \( N \) is weak next: \( X \) requires the existence of a successor state, while \( N \) does not. Thus \( N \phi \) is always true in the last state of a finite trace, since no successor exists there. This distinction is specific to LTL \_f.

LTL \_f formulas are interpreted over finite traces (De Giacomo and Vardi 2013). Given an atom set \( \mathcal{P} \), we define \( \Sigma = 2^\mathcal{P} \) be the family of sets of atoms. Let \( \xi \in \Sigma^+ \) be a finite nonempty trace, with \( \xi = \sigma_0 \sigma_1 \ldots \sigma_n \), we use \( |\xi| = n+1 \) to denote the length of \( \xi \). Moreover, for \( 0 \leq i \leq n \), we denote \( \xi[i] \) as the \( i \)-th position of \( \xi \), and \( \xi_i \) to represent \( \sigma_i \sigma_{i+1} \ldots \sigma_n \), which is the suffix of \( \xi \) from position \( i \). We define the satisfaction relation \( \xi \models \phi \) as follows:

- \( \xi \models tt \); and \( \xi \models p \), if \( p \in \mathcal{P} \) and \( p \in \xi[0] \);
- \( \xi \models \neg \phi \), if \( \xi \not\models \phi \);
- \( \xi \models \phi \land \psi \), if \( \xi \models \phi \) and \( \xi \models \psi \);
- \( \xi \models X\phi \), if \( \xi[i] > 1 \) and \( \xi_1 \models \psi \);
- \( \xi \models (\phi_1 U\phi_2) \), if there exists \( 0 \leq i < |\xi| \) such that \( \xi_i \models \phi_1 \) and for every \( 0 \leq j < i \) it holds that \( \xi_j \models \phi_2 \).

\begin{definition}[LTL \_f Satisfiability Problem] Given an LTL \_f formula \( \phi \) over the alphabet \( \Sigma \), we say \( \phi \) is satisfiable iff there is a finite nonempty trace \( \xi \in \Sigma^+ \) such that \( \xi \models \phi \).
\end{definition}

\begin{notations} We use \( cl(\phi) \) to denote the set of subformulas of \( \phi \). Let \( \mathcal{A} \) be a set of LTL \_f formulas, we denote \( \bigwedge \mathcal{A} \) to be the formula \( \bigwedge_{\phi \in \mathcal{A}} \phi \). The two LTL \_f formulas \( \phi_1, \phi_2 \) are semantically equivalent, denoted as \( \phi_1 \equiv \phi_2 \), iff for every finite trace \( \xi \), \( \xi \models \phi_1 \) iff \( \xi \models \phi_2 \). Obviously, we have \( (\phi_1 \lor \phi_2) \equiv \neg(\neg \phi_1 \land \neg \phi_2) \), \( \mathcal{N} \psi \equiv \neg X \neg \psi \) and \( (\phi_1 R \phi_2) \equiv \neg(\neg \phi_1 U \neg \phi_2) \).
\end{notations}

We say an LTL \_f formula \( \phi \) is in Tail Normal Form (TNF) if \( \phi \) is in Negated Normal Form (NNF) and \( N \)-free. It is trivial to know that every LTL \_f formula has an equivalent NNF. Assume \( \phi \) is in NNF, \( \text{Inf}(\phi) \) is defined as \( t(\phi) \land \mathcal{F} \text{Tail} \), where \( \text{Tail} \) is a new atom to identify the last state of satisfying traces (Motivated from (De Giacomo and Vardi 2013)), and \( t(\phi) \) is an LTL \_f formula defined recursively as follows:

1. \( t(\phi) = \phi \) if \( \phi \) is \( tt, ff \) or a literal;
2. \( t(X\psi) = \neg \text{Tail} \land X(t(\psi)) \);
3. \( t(\mathcal{N} \psi) = \text{Tail} \lor X(t(\psi)) \);
4. \( t(\phi_1 \land \phi_2) = t(\phi_1) \land t(\phi_2) \);
5. \( t(\phi_1 \lor \phi_2) = t(\phi_1) \lor t(\phi_2) \);
6. \( t(\phi_1 U\phi_2) = \neg(\neg \phi_1 \land \neg \phi_2) \).

\begin{theorem} \( \phi \) is satisfiable iff \( \text{Inf}(\phi) \) is satisfiable.
\end{theorem}

In the rest of the paper, unless clearly specified, the input LTL \_f formula is in TNF.

\section*{Approach Overview}

There is a Non-deterministic Finite Automaton (NFA) \( A_\phi \) that accepts exactly the same language as an LTL \_f formula \( \phi \) (De Giacomo and Vardi 2013). Instead of constructing the NFA for \( \phi \), we generate the corresponding transition system (Definition 5), by leveraging SAT solvers. The transition system can be considered as an intermediate structure of the NFA, in which every state consists of a set of subformulas of \( \phi \).

The classic approach to generate the NFA from an LTL \_f formula, i.e. Tableau Construction (Gerth et al. 1995), creates the set of all one-transition next states of the current state. However, the number of these states is extremely large. To mitigate the overload, we leverage SAT solvers to compute the next states of the current state iteratively. Although both approaches share the same worst case (computing all states in the state space), our new approach is better for on-the-fly checking, as it computes new states only if the satisfiability of the formula cannot be determined based on existing states.

We show the SAT-based approach via an example. Consider the formula \( \phi = (\neg \text{Tail} \land \mathcal{A}) \mathcal{U} b \). The initial state \( s_0 \) of the transition system is \( \{\phi\} \). To compute the next states of \( s_0 \), we translate \( \phi \) to its equivalent nExt Normal Form (XNF), e.g. \( \text{XInf}(\phi) = (b \lor ((\neg \text{Tail} \land \mathcal{A}) \land X\phi)) \), see Definition 4. If we replace \( X\phi \) in \( \text{XInf}(\phi) \) with a new propositions \( p_1 \), the new formula, denoted \( \text{XInf}(\phi)p \), is a pure Boolean formula. As a result, a SAT solver can compute an assignment for the formula \( \text{XInf}(\phi)p \). Assume the assignment is \( \{a, \neg b, \neg \text{Tail}, p_1\} \), then we can induce \( \langle a \land b \land \neg \text{Tail} \land X\phi \Rightarrow \phi \rangle \) is true, which indicates \( \{\phi\} = s_0 \) is a one-transition next state of \( s_0 \), i.e. \( s_0 \) has a self-loop with the label \( \{a, \neg b, \neg \text{Tail}\} \). To compute another next state of \( s_0 \), we add the constraint \( \neg p_1 \) to the input of the SAT solver. Repeat the above process and we can construct all states in the transition system.

Checking the satisfiability of \( \phi \) is then reduced to finding a final state (Definition 6) in the corresponding transition system. Since \( \phi \) is in TNF, a final state \( s \) meets the constraint that \( \text{Tail} \land \text{XInf}(\phi)p \) (recall \( s \) is a set of subformulas of \( \phi \)) is satisfiable. For the above example, the initial state \( s_0 \) actually a final state, as \( \text{Tail} \land \text{XInf}(\phi)p \) is satisfiable. Because all states computed by the SAT solver in the transition system are reachable from the initial state, we can prove that \( \phi \) is satisfiable iff there is a final state in the system (Theorem 4).
We present a conflict-driven algorithm, i.e., CDLSC, to accelerate the satisfiability checking. CDLSC maintains a conflict sequence $C$, in which each element, denoted as $C[i]$ $(0 \leq i < |C|)$, is a set of states in the transition system that cannot reach a final state in $i$ steps. Starting from the initial state, CDLSC iteratively checks whether a final state can be reached, and makes use of the conflict sequence to accelerate the search. Consider the formula $\phi = \neg(Tail)\cup(a \wedge \neg(Tail)\cup(b \wedge \neg(Tail)\cup(c)$. In the first iteration, CDLSC checks whether the initial state $s_0 = \{\emptyset\}$ is a final state, i.e., whether $Tail \wedge xnf(\phi)^P$ is satisfiable. The answer is negative, so $s_0$ cannot reach a final state in 0 steps and can be added into $C[0]$. However, we can do better by leveraging the Unsatisfiable Core (UC) returned from the SAT solver. Assume that we get the UC $u_1 = \{\neg(Tail)\cup(a, \neg(Tail)\cup(a)\}$. That indicates every state $s$ containing $u$, i.e., $s \supseteq u$, is not a final state. As a result, we can add $u$ instead of $s_0$ into $C[0]$, making the algorithm much more efficient.

Now in the second iteration, CDLSC first tries to compute a one-transition next state of $s_0$ that is not included in $C[0]$. (Otherwise the new state cannot reach a final state in 0 steps.) This can be encoded as a Boolean formula $xnf(\phi)^P \wedge \neg(p_1 \wedge p_2)$ where $p_1, p_2$ represent $X'((\neg(Tail)\cup(a$ and $X'((\neg(Tail)\cup(a)$ respectively. Assume the new state $s_1 = \{\neg(Tail)\cup(a, \neg(Tail)\cup(b), \neg(Tail)\cup(c)\} is generated from the assignment of the SAT solver. Then CDLSC checks whether $s_1$ can reach a final state in 0 step, i.e. $xnf(s_1)^P \wedge Tail$ is satisfiable. The answer is negative and we can add the UC $u_2 = \{\neg(Tail)\cup(b, \neg(Tail)\cup(b)\} to $C[0]$ as well. Now to compute a next state of $s_0$ that is not included in $C[0]$, the encoded Boolean formula becomes $xnf(\phi)^P \wedge \neg(p_1 \wedge p_2) \wedge \neg(p_3 \wedge p_4)$ where $p_3, p_4$ represent $X'((\neg(Tail)\cup(b)$ and $X'((\neg(Tail)\cup(b)$ respectively. Assume the new state $s_2 = \{\neg(Tail)\cup(a, \neg(Tail)\cup(b), \neg(Tail)\cup(c)\} is generated from the assignment of the SAT solver. Since $xnf(s_2)^P \wedge Tail$ is satisfiable, $s_2$ is a final state and we can conclude that the formula $\phi$ is satisfiable. In principle, there are a total of $2^{|C|} = 32$ states in the transition system of $\phi$, but CDLSC succeeds to find the answer by computing only 3 of them (including the initial state).

CDLSC also leverages the conflict sequence to accelerate checking unsatisfiable formulas. Similar to Bounded Model Checking (BMC) (Biere et al. 1999), CDLSC searches the model iteratively. However, BMC invokes only 1 SAT call for each iteration, while CDLSC invokes multiple SAT calls. CDLSC is more like an IC3-style algorithm, but achieves a much simpler implementation by using UC instead of the Minimal Inductive Core (MIC) like IC3 (Bradley 2011).

### SAT-based Explicit-State Checking

Given an LTL$_f$ formula $\phi$, we construct the LTL$_f$ transition system (Li et al. 2014; 2015) by SAT solvers and then check the satisfiability of the formula over its corresponding transition system.

### LTL$_f$ Transition System

First, we show how one can consider LTL$_f$ formulas as propositional ones. This requires considering temporal subformulas as propositional atoms.

#### Definition 2 (Propositional Atoms)

For an LTL$_f$ formula $\phi$, we define the set of propositional atoms of $\phi$, i.e., PA($\phi$), as follows: (1) $PA(\phi) = \{\phi\}$ if $\phi$ is an atom, Next, Until or Release formula; (2) $PA(\phi) = PA(\psi)$ if $\phi = (\exists \psi)$; (3) $PA(\phi) = \bigcup PA(\phi_1) \cup PA(\phi_2)$ if $\phi = (\phi_1 \land \phi_2)$ or $\phi = (\phi_1 \lor \phi_2)$.

Consider $\phi = (a \land ((\neg(Tail) \land a)\cup b) \land (\neg(Tail) \land X(a \lor b))$. We have $PA(\phi) = \{a, Tail, (\neg(Tail) \land a)\cup b, (X(a \lor b))\}$. Intuitively, the propositional atoms are obtained by treating all temporal subformulas of $\phi$ as atomic propositions. Thus, an LTL$_f$ formula $\phi$ can be viewed as a propositional formula over PA($\phi$).

#### Definition 3

For an LTL$_f$ formula $\phi$, let $\phi^P$ be $\phi$ considered as a propositional formula over PA($\phi$). A propositional assignment $A$ of $\phi^P$, is in $2^{PA(\phi)}$ and satisfies $A \models \phi^P$.

Consider the formula $\phi = (\exists a, (\neg(Tail) \land a)\cup b) \land (\exists (\neg(Tail) \land c)\cup d)$. From Definition 3, $\phi^P$ is $(\exists a \land p_1) \land (b \lor (\exists (\neg(Tail) \land c)\cup d))$ where $p_1, p_2$ are two Boolean variables representing the truth values of $(\neg(Tail) \land a)\cup b$ and $(\exists (\neg(Tail) \land c)\cup d)$. Moreover, the set $\{a, p_1, (\neg(Tail) \land a)\cup b, b, -p_2((\exists (\neg(Tail) \land c)\cup d))$ is a propositional assignment of $\phi^P$. In the rest of the paper, we do not introduce the intermediate variables and directly say $\{\exists a, (\neg(Tail) \land a)\cup b, b, -p_2((\exists (\neg(Tail) \land c)\cup d))$ is a propositional assignment of $\phi^P$. The following theorem shows the relationship between the propositional assignment of $\phi^P$ and the satisfaction of $\phi$.

#### Theorem 2

For an LTL$_f$ formula $\phi$ and a finite trace $\xi$, $\xi \models \phi$ implies there exists a propositional assignment $A$ of $\phi^P$ such that $\xi \models A$; On the other hand, $\xi \models \bigwedge A$ where $A$ is a propositional assignment of $\phi^P$, also implies $\xi \models \phi$.

We now introduce the neXt Normal Form (XNF) of LTL$_f$ formulas, which is useful for the construction of the transition system.

#### Definition 4 (neXt Normal Form)

An LTL$_f$ formula $\phi$ is in neXt Normal Form (XNF) if there are no Until or Release subformulas of $\phi$ in PA($\phi$).

For example, $\phi = ((\neg(Tail) \land a)\cup b)$ is not in XNF, while $(b \lor (\neg(Tail) \land a)\cup X(\exists a(\neg(Tail) \land a)\cup b))$ is. Every LTL$_f$ formula $\phi$ has a linear-time conversion to an equivalent formula in XNF, which we denoted as $xnf(\phi)$.

#### Theorem 3

For an LTL$_f$ formula $\phi$, there is a corresponding LTL$_f$ formula $xnf(\phi)$ in XNF such that $\phi \equiv xnf(\phi)$. Furthermore, the cost of the conversion is linear.

Observe that when $\phi$ is in XNF, there can be only Next (no Until or Release) temporal formulas in the propositional assignment of $\phi^P$. For $\phi = b \land (a \land (a \land Tail))$, the set $A = \{a, b, \neg(Tail), X(a\cup b)\}$ is a propositional assignment of $\phi^P$. Based on LTL$_f$ semantics, we can induce from $A$ that if a finite trace $\xi$ satisfying $\xi \models A$ \geq \{a, b, \neg(Tail)\}$ and $\xi \models \phi$ is true. This motivates us to construct the transition system for $\phi$, in which $a\cup b$ is a next state of $\phi$ and $\{a, b, \neg(Tail)\}$ is the transition label between these two states.
Let $\phi$ be an LTL$_f$ formula and $A$ be a propositional assignment of $\phi^p$, we denote $L(A) = \{ l \mid l \in A \text{ is a literal} \}$ and $X(A) = \{ \theta \mid X\theta \in A \}$. Now we define the transition system for an LTL$_f$ formula.

**Definition 5.** Given an LTL$_f$ formula $\phi$ and its literal set $\mathcal{L}$, let $\Sigma = 2^\mathcal{L}$. We define the transition system $T_\phi = (S, s_0, T)$ for $\phi$, where $S \subseteq 2^{e(\phi)}$ is the set of states, $s_0 = \{ \phi \} \in S$ is the initial state, and

- $T : S \times \Sigma \rightarrow 2^S$ is the transition relation, where $s_2 \in T(s_1, \sigma)$ ($\sigma \in \Sigma$) holds iff there is a propositional assignment $A$ of $x\neg\neg(\wedge S_1)^p$ such that $\sigma \supseteq L(A)$ and $s_2 = X(A)$.

A run of $T_\phi$ on a finite trace $\xi([\xi] = n > 0)$ is a finite sequence $s_0, s_1, \ldots, s_n$ such that $s_0$ is the initial state and $s_{i+1} \in T(s_i, \xi[i])$ holds for all $0 \leq i < n$.

We define the notation $|r|$ for a run $r$, to represent the length of $r$, i.e. number of states in $r$. We say state $s_2$ is reachable from state $s_1$ in $i$ ($i \geq 0$) steps (resp. in up to $i$ steps), if there is a run $r$ on some finite trace $\xi$ leading from $s_1$ to $s_2$ and $|r| = i$ (resp. $|r| \leq i$). In particular, we say $s_2$ is a one-transition next state of $s_1$ if $s_2$ is reachable from $s_1$ in 1 steps. Since a state $s$ is a subset of $c(\phi)$, which essentially is a formula with the form of $\wedge_{\psi \in \phi} \psi$, we mix the usage of the state and formula in the rest of the paper. That is, a state can be a formula of $\wedge_{\psi \in \phi} \psi$, and a formula $\phi$ can be a set of states, i.e. $s \in \phi$ iff $s \Rightarrow \phi$.

**Lemma 1.** Let $T_\phi = (S, s_0, T)$ be the transition system of $\phi$. Every state $s \in S$ is reachable from the initial state $s_0$.

**Definition 6** (Final State). Let $s$ be a state of a transition system $T_\phi$. Then $s$ is a final state of $T_\phi$ iff the Boolean formula $T(\lnot \wedge X\neg\neg(S))^p$ is satisfiable.

By introducing the concept of final state, we are able to check the satisfiability of the LTL$_f$ formula $\phi$ over $T_\phi$.

**Theorem 4.** Let $\phi$ be an LTL$_f$ formula. Then $\phi$ is satisfiable iff there is a final state in $T_\phi$.

An intuitive solution from Theorem 4 to check the satisfiability of $\phi$ is to construct states of $T_\phi$ until (1) either a final state is found by Definition 6, meaning $\phi$ is satisfiable; or (2) all states in $T_\phi$ are generated but no final state can be found, meaning $\phi$ is unsatisfiable. This approach is simple and easy to implement, however, it does not perform well according to our preliminary experiments.

**Conflict-Driven LTL$_f$ Satisfiability Checking**

In this section, we present a conflict-driven algorithm for LTL$_f$ satisfiability checking. The new algorithm is inspired by (Li et al. 2015), where information of both satisfiability and unsatisfiability results of SAT solvers are used. The motivation is as follows: In Definition 6, if the Boolean formula $T(\lnot \wedge X\neg\neg(S))^p$ is unsatisfiable, the SAT solver is able to provide a UC (Unsatisfiable Core) $c$ such that $c \subseteq s$ and $T(\lnot \wedge X\neg\neg(C))^p$ is still unsatisfiable. It means that $c$ represents a set of states that are not final states. By adding a new constraint $\neg(\wedge_{\psi \in c} C\psi)$, the SAT solver can provide a model (if exists) that avoids negation of those states in $c$, which accelerates the search of final states. More generally, we define the conflict sequence, which is used to maintain all information of UCs acquired during the checking process.

**Definition 7** (Conflict Sequence). Given an LTL$_f$ formula $\phi$, a conflict sequence $C$ for the transition system $T_\phi$ is a finite sequence of set of states such that:

1. The initial state $s_0 = \{ \phi \}$ is in $C[i]$ for $0 \leq i < |C|$;
2. Every state in $C[0]$ is not a final state;
3. For every state $s \in C[i + 1] \setminus \{ 0 \leq i < |C| - 1 \}$, all the one-transition next states of $s$ are included in $C[i]$.

We call each $C[i]$ a frame, and $i$ is the frame level.

In the definition, $|C|$ represents the length of $C$ and $C[i]$ denotes the $i$-th element of $C$. Consider the transition system shown in Figure 1, in which $s_0$ is the initial state and $s_4$ is the final state. Based on Definition 7, the sequence $C = \{ s_0, s_1, s_2, s_3 \}, \{ s_0, s_1 \}, \{ s_0 \}$ is a conflict sequence. Notably, the conflict sequence for a transition system may not be unique. For the above example, the sequence $\{ s_0, s_1 \}, \{ s_0 \}$ is also a conflict sequence for the system. This suggests that the construction of a conflict sequence is algorithm-specific. Moreover, it is not hard to induce that every non-empty prefix of a conflict sequence is also a conflict sequence. For example, a prefix of $C$ above, i.e. $\{ s_0, s_1, s_2, s_3 \}, \{ s_0, s_1 \}$, is a conflict sequence. As a result, a conflict sequence can be constructed iteratively, i.e. the elements can be generated (and updated) in order. Our new algorithm is motivated by these two observations.

![Figure 1: An example transition system for the conflict sequence.](image-url)
We are able to utilize the conflict sequence to accelerate the satisfiability checking of LTLf formulas, using the theoretical foundations provided by Theorem 5 and 6 below.

**Theorem 5.** The LTLf formula \( \phi \) is satisfiable iff there is a run \( r = s_0, s_1, \ldots, s_n (n \geq 0) \) of \( T_\phi \) such that (1) \( s_0 \) is a final state; and (2) \( s_i \) (\( 0 \leq i \leq n \)) is not in \( C[n-i] \) for every conflict sequence \( C \) of \( T_\phi \) with \( |C| > n-i \).

**Proof.** (\( \Rightarrow \)) Since \( s_n \) is a final state, \( \phi \) is satisfiable according to Theorem 4. (\( \Leftarrow \)) Since \( \phi \) is satisfiable, there is a finite trace \( \xi \) such that the corresponding run \( r \) of \( T_\phi \) on \( \xi \) ends with a final state (according to Theorem 4). Let \( r \) be \( s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_n \), where \( s_n \) is the final state. It holds that \( s_i \) (\( 0 \leq i \leq n \)) is a state that can reach a final state in \( n-i \) steps. Moreover for every \( C \) of \( T_\phi \) with \( |C| > n-i \), \( C[n-i] \) is meaningless when \( |C| \leq n-i \) represents a set of states that cannot reach a final state in \( n-i \) steps (From the proof of Lemma 2). As a result, it is true that \( s_i \) is not in \( C[n-i] \) if \( |C| > n-i \).

Theorem 5 suggests that to check whether a state \( s \) can reach a final state in \( i \) steps (\( i \geq 1 \)), finding a one-transition next state \( s' \) of \( s \) that is not in \( C[i-1] \) is necessary; as \( s' \in C[i-1] \) implies \( s' \) cannot reach a final state in \( i-1 \) steps (From the proof of Lemma 2). If all one-transition next states of \( s \) are in \( C[i-1] \), \( s \) cannot reach a final state in \( i \) steps.

**Theorem 6.** The LTLf formula \( \phi \) is unsatisfiable iff there is a conflict sequence \( C \) and \( i \geq 0 \) such that \( \bigcap_{0 \leq j \leq i} C[j] \subseteq C[i+1] \).

**Proof.** (\( \Leftarrow \)) \( \bigcap_{0 \leq j \leq i} C[j] \subseteq C[i+1] \) is true implies that \( \bigcap_{0 \leq j \leq i} C[j] = \bigcap_{0 \leq j \leq i+1} C[j] \) is true. Also from Lemma 2 we know \( \bigcap_{0 \leq j \leq i} C[j] \) is a set of states that cannot reach a final state in up to \( i \) steps. Since \( \phi \in C[i] \) is true for each \( i \geq 0 \), \( \phi \) is in \( \bigcap_{0 \leq j \leq i} C[j] \). Moreover, \( \bigcap_{0 \leq j \leq i} C[j] = \bigcap_{0 \leq j \leq i+1} C[j] \) is true implies all reachable states from \( \phi \) are included in \( \bigcap_{0 \leq j \leq i} C[j] \). We have known all states in \( \bigcap_{0 \leq j \leq i} C[j] \) are not final states, so \( \phi \) is unsatisfiable.

(\( \Rightarrow \)) If \( \phi \) is unsatisfiable, every state in \( T_\phi \) is not a final state. Let \( S \) be the set of states of \( T_\phi \). According to Lemma 2, \( \bigcap_{0 \leq j \leq i} C[j] (i \geq 0) \) contains the set of states that are not final in up to \( i \) steps. Now we let \( C \) satisfy that \( \bigcap_{0 \leq j \leq i} C[j] (i \geq 0) \) contains all states that are not final in up to \( i \) steps, so \( \bigcap_{0 \leq j \leq i} C[j] \) includes all reachable states from \( \phi \), as \( \phi \) is unsatisfiable. However, because \( \bigcap_{0 \leq j \leq i} C[j] \supseteq \bigcap_{0 \leq j \leq i+1} C[j] \supseteq S(i \geq 0) \), there must be an \( i \geq 0 \) such that \( \bigcap_{0 \leq j \leq i} C[j] = \bigcap_{0 \leq j \leq i+1} C[j] \), which indicates that \( \bigcap_{0 \leq j \leq i} C[j] \subseteq C[i+1] \) is true.

**Algorithm Design.** The algorithm, named CDLSC (Conflict-Driven LTLf Satisfiability Checking), constructs the transition system on-the-fly. The initial state \( s_0 \) is fixed to be \( \{ \phi \} \) where \( \phi \) is the input formula. From Definition 6, whether a state \( s \) is final is reducible to the satisfiability checking of the Boolean formula \( Tail \wedge \text{xnf}(s)^p \). If \( s_0 \) is a final state, there is no need to maintain the conflict sequence in CDLSC, and the algorithm can return SAT immediately; Otherwise, the conflict sequence is maintained as follows.

- In CDLSC, every element of \( C \) is a set of set of subformulas of the input formula \( \phi \). Formally, each \( C[i] (i \geq 0) \) can be represented by the LTLf formula \( \bigvee_{e \in C[i]} \bigwedge_{\psi \in e} \psi \) where \( e \) is a set of subformulas of \( \phi \). We mix-use the notation \( C[i] \) for the corresponding LTLf formula as well.
- Every state \( s \) satisfying \( s \Rightarrow C[i] \) is included in \( C[i] \).
- \( C \) is created iteratively. In each iteration \( i \geq 0 \), \( C[i] \) is initialized as the empty set.
- To compute elements in \( C[0] \), we consider an existing state \( s \) (e.g. \( s_0 \)). If the Boolean formula \( Tail \wedge \text{xnf}(s)^p \) is unsatisfiable, \( s \) is not a final state and can be added into \( C[0] \) from Item 2 of Definition 7. Moreover, CDLSC leverages the Unsatisfiable Core (UC) technique from the SAT community to add a set of states, all of which are not final and include \( s \), to \( C[0] \). This set of states, denoted as \( c \), is also represented by a set of LTLf formulas and satisfies \( c \subseteq s \). The detail to obtain \( c \) is discussed below.
- To compute elements in \( C[i+1] \) (\( i \geq 0 \)), we consider the Boolean formula \( \text{xnf}(s) \wedge \neg A' (C[i]) \). We use the formula \( \text{xnf}(s) \wedge \neg A' (C[i]) \) to check whether a one-transition next state of \( s \) is not in \( C[i] \). If the formula is unsatisfiable, all the one-transition next states of \( s \) are in \( C[i] \), thus \( s \) can be added into \( C[i+1] \) according to Item 3 of Definition 7. Similarly, we also utilize the UC technique to obtain a subset \( c \) of \( s \), such that \( c \) represents a set of states that can be added into \( C[i+1] \).

As shown above, every Boolean formula sent to a SAT solver has the form of \( (\text{xnf}(s) \wedge \theta)^p \) where \( s \) is a state and \( \theta \) is either \( Tail \) or \( \neg A' (C[i]) \). Since every state \( s \) consists of a set of LTLf formulas, the Boolean formula can be rewritten as \( \alpha_1 = (\bigwedge_{\psi \in s} \text{xnf}(\psi) \wedge \theta)^p \). Moreover, we introduce a new variable \( p_\psi \) for each \( \psi \in s \), and re-encode the formula to be \( \alpha_2 = \bigwedge_{\psi \in s} p_\psi \wedge (\bigwedge_{\psi \in s} \text{xnf}(\psi) \vee \neg p_\psi) \wedge \theta)^p \). \( \alpha_3 \) is satisfiable iff \( \alpha_1 \) is satisfiable, and \( A \) is an assignment of \( \alpha_2 \) if \( \{ \psi \mid p_\psi \in \psi \} \) is an assignment of \( \alpha_1 \). Sending \( \alpha_2 \) instead of \( \alpha_1 \) to the SAT solver that supports assumptions (e.g. Minisat (Één and Sörensson 2003)) enables the SAT solver to return the UC, which is a set of states, when \( \alpha_2 \) is unsatisfiable. For example, assume \( s = \{ \psi_1, \psi_2, \psi_3 \} \) and \( \alpha_2 \) is sent to the SAT solver with \( \{ \psi_\psi_i \mid i \in \{1, 2, 3\} \} \) being the assumptions. If the SAT solver returns unsatisfiable and the UC \( \{ p_\psi \} \), the set \( c = \{ \psi_1 \} \), which represents every state including \( \psi_1 \), is the one to be added into the corresponding \( C[i] \). We use the notation \( \text{get_uc}() \) for the above procedure.

The pseudo-code of CDLSC is shown in Algorithm 1. Line 1-2 considers the situation when the input formula \( \phi \) is a final state itself. Otherwise, the first frame \( C[0] \) is initialized to \( \{ \phi \} \) (Line 3), and the current frame level is set to 0 (Line 4). After that, the loop body (Line 5-11) keeps updating the elements of \( C \) iteratively, until either the procedure \( \text{try_satisfy}() \) returns true, which means to find a model of \( \phi \), or the procedure \( \text{inv_found}() \) returns true, which is the implementation of Theorem 6. The loop continues to create a new frame in \( C \) if neither of the procedures succeeds to return true. To describe conveniently, we say every run of the while loop body in Algorithm 1 is an iteration.
Algorithm 1 Implementation of CDLSC

Input: An LTL$_f$ formula $\phi$.
Output: SAT or UNSAT.
1: if $\text{Tail} \land \text{xnf}(\phi)^p$ is satisfiable then
2: return SAT;
3: Set $C[0] := \{\phi\}$;
4: Set $\text{frame\_level} := 0$;
5: while true do
6: if $\text{try\_satisfy}(\phi, \text{frame\_level})$ returns true then
7: return SAT;
8: if $\text{inv\_found}($frame\_level)$ returns true then
9: return UNSAT;
10: $\text{frame\_level} := \text{frame\_level} + 1$;
11: Set $C[$frame\_level$] = \emptyset$;

The procedure $\text{try\_satisfy}$ is responsible for updating $C$. Taking an formula $\phi$ and the frame level frame\_level currently working on, $\text{try\_satisfy}$ returns true if a model of $\phi$ can be found, with the length of frame\_level + 1. As shown in Algorithm 2, $\text{try\_satisfy}$ is implemented in a recursive way. Each time it checks whether a next state of the input $\phi$, which belongs to a lower level (than the input frame\_level) frame can be found (Line 2). If the result is positive and such a new state $\phi'$ is constructed, $\text{try\_satisfy}$ first checks whether $\phi'$ is a final state when frame\_level is 0 (in which case returns true). If $\phi'$ is not a final state, a UC is extracted from the SAT solver and added into $C[0]$ (Line 5-11). If frame\_level is not 0, $\text{try\_satisfy}$ recursively checks whether a model of $\phi'$ can be found with the length of frame\_level (Line 12-13). If the result is negative and such a state cannot be constructed, a UC is extracted from the SAT solver and added into $C[\text{frame\_level} + 1]$ (Line 14-15).

Algorithm 2 Implementation of $\text{try\_satisfy}$

Input: $\phi$: The formula is working on; frame\_level: The frame level is working on.
Output: true or false.
1: Let $\psi = \neg\forall(C[\text{frame\_level}])$;
2: while $(\psi \land \text{xnf}(\phi))^p$ is satisfiable do
3: Let $A$ be the model of $(\psi \land \text{xnf}(\phi))^p$;
4: Let $\phi' = X(A)$, i.e. the next state of $\phi$ extracted from $A$;
5: if frame\_level == 0 then
6: if $\text{Tail} \land \text{xnf}(\phi')^p$ is satisfiable then
7: return true;
8: else
9: let $c = \text{get\_uc}()$;
10: Add $c$ into $C[$frame\_level$]$;
11: Continue;
12: if $\text{try\_satisfy}(\phi', \text{frame\_level} - 1)$ is true then
13: return true;
14: Let $c = \text{get\_uc}()$;
15: Add $c$ into $C[\text{frame\_level} + 1]$;
16: return false;

Notably, Item 1 of Definition 7, i.e. $\{\phi\} \in C[i]$, is guaranteed for each $i \geq 0$, as the original input formula of $\text{try\_satisfy}$ is always $\phi$ (Line 6 in Algorithm 1) and there is some $c$ (Line 15 in Algorithm 2) including $\{\phi\}$ that is added into $C[i]$, if no model can be found in the current iteration.

The procedure $\text{inv\_found}$ in Algorithm 1 implements Theorem 6 in a straightforward way: It reduces the checking of whether $\bigwedge_{0 \leq j \leq i} C[j] \subseteq C[i + 1]$ being true on some frame level $i$, to the satisfiability checking of the Boolean formula $\bigwedge_{0 \leq j \leq i} C[j] \Rightarrow \{i + 1\}$. Finally, we state Theorem 7 below to provide the theoretical guarantee that CDLSC always terminates correctly.

Lemma 3. After each iteration of CDLSC with no model found, the sequence $C$ is a conflict sequence of $T_\psi$ for the transition system $T_\psi$.

Theorem 7. The CDLSC algorithm terminates with a correct result.

CDLSC is shown how to accelerate the checking of satisfiable formulas in the previous section. For unsatisfiable instances, consider $\phi = (\neg\text{Tail})\{a \land (\text{Tail})\neg a \land (\neg\text{Tail})\{b\}$. CDLSC first checks that $\text{Tail} \land \text{xnf}(\phi)^p$ is unsatisfiable, where the SAT solver returns $c = \{\neg\text{Tail}\{a\}, \text{Tail}\neg a\}$ as the UC. So $c$ is added into $C[0]$. Then CDLSC checks that $\{\text{xnf}(\phi) \land \neg\forall(C[0])\}^p$ is still unsatisfiable, in which $c = \{\neg\text{Tail}\{a\}, \text{Tail}\neg a\}$ is still the UC. So $c$ is added into $C[1]$ as well. Since $C[0] \subseteq C[1]$ and according to Theorem 6, CDLSC terminates with the unsatisfiable result. In this case, CDLSC only visits one state for the whole checking process. For a more general instance like $\phi \land \psi$, where $\psi$ is a large LTL$_f$ formula, checking by CDLSC enables to achieve a significantly improvement compared to the checking by traditional tableau approach.

Summarily, CDLSC is a conflict-driven on-the-fly satisfiability checking algorithm, which successfully leads to either an earlier finding of a satisfying model, or the faster termination with the unsatisfiable result.

Experimental Evaluation

Benchmarks We first consider the LTL-as-LTL$_f$ benchmark, which is evaluated by previous works on LTL$_f$ satisfiability checking (Li et al. 2014; Fionda and Greco 2016). This benchmark consists of 7442 instances that are originally LTL formulas but are treated as LTL$_f$ formulas, as both logics share the same syntax. Previous works (Li et al. 2014; Fionda and Greco 2016) have shown that the benchmark is useful to test the scalability of LTL$_f$ solvers.

Secondly, we consider the 7 LTL$_f$-specific patterns that are introduced in recent researches on LTL$_f$, e.g. (De Giacomo, Maselli, and Montali 2014; Di Ciccio, Maggi, and Mendling 2016), and we create 100 instances for each pattern. As shown in Table 1, it is trivial to check the satisfiability of these LTL$_f$ patterns by most tested solvers, as either they have small sizes or dedicated heuristics for LTL$_f$, which are encoded in both AltaLTL-finite and CDLSC, enable to solve them quickly. Inspired from the observation in (Li et al. 2013) that an LTL specification in practice is often the conjunction of a set of small and frequently-used patterns, we randomly choose a subset of the instances of the 7
patterns to imitate a real LTL \(_f\) specification in practice. We generate 1000 such instances as the practical conjunction pattern shown in the last row of Table 1. Unlike the random benchmarks in SAT community, which are often considered not interesting, we argue that the new practical conjunction pattern is a representative for real LTL \(_f\) specifications in industry.

**Experimental Setup** We implement CDLSC in the tool aaltaf\(^1\) and use Minisat 2.2.0 (Eén and Sörensson 2003) as the SAT engine. We compare it with two extant LTL \(_f\) satisfiability solvers: Aalta-finite (Li et al. 2014) and ltl2sat (Fionda and Greco 2016). We also compared with the state-of-art LTL solver Aalta-infinite (Li et al. 2015), using the LTL \(_f\)-to-LTL satisfiability-preserving reduction described in (De Giacomo and Vardi 2013). As LTL satisfiability checking is reducible to model checking, as described in (Rozier and Vardi 2007), we also compared with this reduction, using nuXmv with the IC3+K-LIVE back-end (Cavada et al. 2014), as an LTL \(_f\) satisfiability checker.

We ran the experiments on a RedHat 6.0 cluster with 2304 processor cores in 192 nodes (12 processor cores per node), running at 2.83 GHz with 48GB of RAM per node. Each tool executed on a dedicated node with a timeout of 60 seconds, measuring execution time with Unix time. Excluding timeouts, all solvers found correct verdicts for all formulas. All artifacts are available in the supplemental material.

**Results** Figure 2 shows the results for LTL \(_f\) satisfiability checking on LTL-as-LTL \(_f\) benchmarks. CDLSC outperforms all other approaches. On average, CDLSC performs about 4 times faster than the second-best approach IC3+K-LIVE (1705 seconds vs. 6075 seconds). CDLSC checks the LTL \(_f\) formula directly, while IC3+K-LIVE must take the input of the LTL formula translated from the LTL \(_f\) formula. As a result, IC3-KLIVE may take extra cost, e.g. finding a satisfying lasso for the model, to the satisfiability checking. Meanwhile, CDLSC can benefit from the heuristics dedicated for LTL \(_f\) that are proposed in (Li et al. 2014). Finally, the performance of ltl2sat is highly tied to its performance of unsatisfiability checking as most of the timeout cases for ltl2sat are unsatisfiable. For Aalta-finite, its performance is restricted by the heavy cost of Tableau Construction.

Table 1 shows the results for LTL \(_f\)-specific experiments. Columns 1-3 show the types of LTL \(_f\) formulas under test, the number of test instances for each formula type, and the results by formula type. Columns 4-8 show the checking times by formula types in seconds. The dedicated BMC is highly outperformed all other solvers. For Aalta-finite, its performance limits the scalability of the methodology.

<table>
<thead>
<tr>
<th>Type</th>
<th>Number</th>
<th>Result</th>
<th>IC3+K-LIVE</th>
<th>Aalta-finite</th>
<th>Aalta-infinite</th>
<th>ltl2sat</th>
<th>CDLSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternate Response</td>
<td>100</td>
<td>sat</td>
<td>134</td>
<td>1</td>
<td>48</td>
<td>123</td>
<td>3</td>
</tr>
<tr>
<td>Alternate Precedence</td>
<td>100</td>
<td>sat</td>
<td>154</td>
<td>3</td>
<td>70</td>
<td>380</td>
<td>4</td>
</tr>
<tr>
<td>Chain Precedence</td>
<td>100</td>
<td>sat</td>
<td>127</td>
<td>2</td>
<td>45</td>
<td>83</td>
<td>2</td>
</tr>
<tr>
<td>Chain Response</td>
<td>100</td>
<td>sat</td>
<td>79</td>
<td>1</td>
<td>41</td>
<td>49</td>
<td>2</td>
</tr>
<tr>
<td>Precedence</td>
<td>100</td>
<td>sat</td>
<td>132</td>
<td>2</td>
<td>14</td>
<td>124</td>
<td>1</td>
</tr>
<tr>
<td>Responded Existence</td>
<td>100</td>
<td>sat</td>
<td>130</td>
<td>1</td>
<td>14</td>
<td>327</td>
<td>1</td>
</tr>
<tr>
<td>Response</td>
<td>100</td>
<td>sat</td>
<td>155</td>
<td>1</td>
<td>41</td>
<td>53</td>
<td>2</td>
</tr>
<tr>
<td>Practical Conjunction</td>
<td>1000</td>
<td>varies</td>
<td>1669</td>
<td>19564</td>
<td>4443</td>
<td>20477</td>
<td>115</td>
</tr>
</tbody>
</table>

Figure 2: Result for LTL \(_f\) Satisfiability Checking on LTL-as-LTL \(_f\) Benchmarks. The X axis represents the number of benchmarks, and the Y axis is the accumulated checking time (s).

**Discussion and Concluding Remarks**

Bounded Model Checking (BMC) (Biere et al. 1999) is also a popular SAT-based technique, which is however, not necessary to compare. There are two ways to apply BMC to LTL \(_f\) satisfiability checking. The first one is to check the satisfiability of the LTL formula from the input LTL \(_f\) formula. (Li et al. 2015) has shown that this approach cannot perform better than IC3+K-LIVE, and the fact of CDLSC outperforming IC3+K-LIVE induces CDLSC also outperforms BMC. The second approach is to check the satisfiability of the LTL \(_f\) formula \(\phi\) directly, by unrolling \(\phi\) iteratively. In the worst case, BMC can terminate (with UNSAT) once the iteration reaches the upper bound. This is exactly what is implemented in ltl2sat (Fionda and Greco 2016).

In this paper, we introduce a new SAT-based framework, based on which we present a conflict-driven algorithm CDLSC, for LTL \(_f\) satisfiability checking. Our experiments demonstrate that CDLSC outperforms Aalta-infinite and IC3+K-LIVE, which are designed for LTL satisfiability checking, showing the advantage of a dedicated algorithm for LTL \(_f\). Notably, CDLSC maintains a conflict sequence, which is similar to the state-of-art model checking technique.

---

\(^{1}\)https://github.com/lijwen2748/aaltaf
IC3 (Bradley 2011). CDLSC does not require the conflict sequence to be monotone, and simply use the UC from SAT solvers to update the sequence. Meanwhile, IC3 requires the sequence to be strictly monotone, and has to compute its dedicated MIC (Minimal Inductive Core) to update the sequence. We conclude that CDLSC outperforms other existing approaches for LTL j satisfiability checking.

Acknowledgement. We thank anonymous reviewers for helpful comments. This work is supported by NASA ECF NNX16AR57G, NSF CAREER Award CNS-1552934, NSF grants CCF-1319459 and IIS-1527668, NSF Expeditions in Computing project “ExCAPE: Expeditions in Computer Augmented Program Engineering”, NSFC projects No. 6157297, 61632005, 61532019, and China HGJ project No. 2017ZX01038102-002.

References


